Exact Solution of a Monomer-Dimer Problem: A Single Boundary Monomer on a Non-Bipartite Lattice

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Abstract

We solve the monomer-dimer problem on a non-bipartite lattice, the simple quartic lattice with cylindrical boundary conditions, with a single monomer residing on the boundary. Due to the non-bipartite nature of the lattice, the well-known method of a Temperley bijection of solving single-monomer problems cannot be used. In this paper we derive the solution by mapping the problem onto one on close-packed dimers on a related lattice. Finite-size analysis of the solution is carried out. We find from asymptotic expansions of the free energy that the central charge in the logarithmic conformal field theory assumes the value $c = -2$.

I. INTRODUCTION

An outstanding unsolved problem in lattice statistics is the monomer-dimer problem. In this problem diatomic molecules adsorbed on a surface are modeled as rigid dimers occupying two adjacent sites and lattice sites not covered by dimers are regarded as occupied by monomers. While the case of pure dimers has been solved in 1961 by Kasteleyn [1] and by Fisher and Temperley [2, 3], the general monomer-dimer problem has proven to be computationally intractable [4].

In 1974, Temperley [5] introduced an intriguing bijection mapping the dimer problem with a single monomer at the corner of a finite $M \times N$ lattice to the counting problem of spanning trees on a related lattice, thereby providing an alternate way of deducing the solution. The method of Temperley bijection has since been extended to the case when the monomer resides on other specific boundary sites [6]. However, the success of the Temperley bijection apparently relies on the fact that the lattices being bipartite; it does not work for non-bipartite lattices. In this paper, we consider one non-bipartite lattice, a rectangular lattice with a cylindrical boundary condition. By using an alternate mapping formulated recently by one of us [7, 8], we solve the monomer-dimer problem on this lattice when a single monomer resides on the boundary. We also clarify the mathematical content of the solution by carrying out finite-size analysis of the solution.

II. SINGLE MONOMER ON THE BOUNDARY OF A CYLINDER

Consider a simple quartic lattice $\mathcal{L}$ consisting of an array of N rows and M columns embedded on the surface of a cylinder with periodic boundary conditions imposed in the horizontal direction. See Fig. 1(a) for an illustration. For MN odd, hence both M, N odd, the lattice is not bipartite. But the lattice can be fully covered by one monomer and $(MN - 1)/2$ dimers. We consider the problem of evaluating its generating function when the single monomer resides on the boundary.

On first sight, one would attempt to use the Temperley bijection of mapping. However, it can be readily verified that the attempt invariably fails, apparently due to the fact that $\mathcal{L}$ is not bipartite. Instead, we adopt an alternate formulation devised by one of us [7, 8] which does not make use of the Temperley bijection.
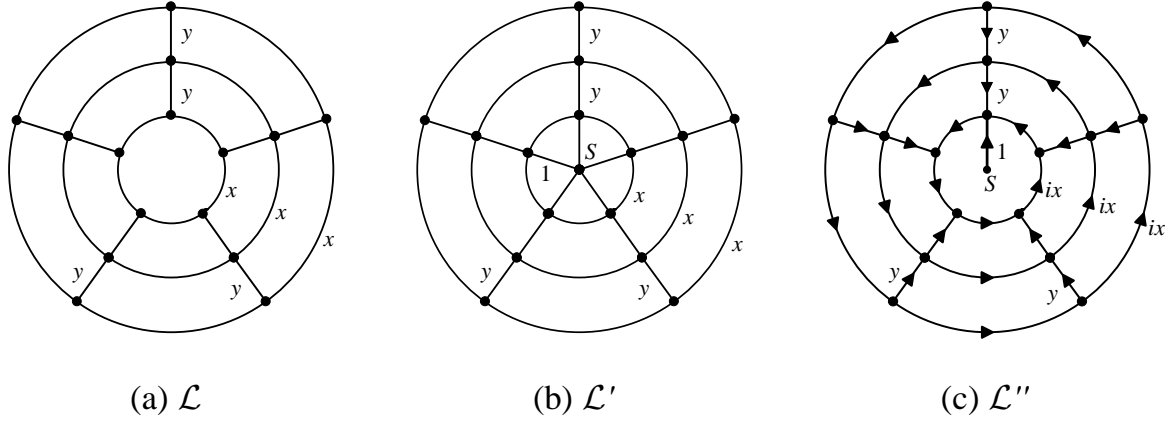
FIG. 1: (a) A simple quartic lattice $\mathcal{L}$ consisting of an array of $N = 3$ rows and $M = 5$ columns embedded on the surface of a cylinder. (b) A self-dual lattice $\mathcal{L}'$ derived from $\mathcal{L}$ by adding a new site S connected to all M sites of one boundary of $\mathcal{L}$. (c) A oriented lattice $\mathcal{L}''$ constructed from $\mathcal{L}'$ by keeping only one edge connecting to S. A phase factor i is associated to all x dimers.

Denote the desired generating function by

$$G_{MD}(x, y) = \sum_{\text{config}} x^{n_1} y^{n_2}, \tag{1}$$

where the summation runs over all monomer-dimer configurations with a single monomer on one of the two boundaries, $x > 0$ and $y > 0$ are the weights of, respectively, horizontal and vertical dimers as indicated in Fig. 1(a), and n_1 and n_2 the numbers of horizontal and vertical dimers subject to $n_1 + n_2 = (MN - 1)/2$. For quick reference we first give the final result which holds for $M, N \geq 3$,

$$G_{MD}(x, y) = 2Mx^{(M-1)/2}y^{(N-1)/2} \prod_{m=1}^{\frac{M-1}{2}} \prod_{n=1}^{\frac{N-1}{2}} \left(4x^2 \sin^2 \frac{2m\pi}{M} + 4y^2 \cos^2 \frac{n\pi}{N+1} \right). \tag{2}$$

In contrast, the monomer-dimer generating function with a single monomer on the boundary of an $M \times N$ net with free (open) boundaries is [6]

$$G_{MD}^{\text{free}}(x, y) = (M + N - 2)x^{(M-1)/2}y^{(N-1)/2} \prod_{m=1}^{\frac{M-1}{2}} \prod_{n=1}^{\frac{N-1}{2}} \left(4x^2 \cos^2 \frac{m\pi}{M+1} + 4y^2 \cos^2 \frac{n\pi}{N+1} \right), \tag{3}$$

where the factor $M + N - 2$ is the number of equivalent boundary sites where the monomer can reside.

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Results of enumerations of (2) and (3) for small lattices are shown in Table I.

TABLE I: Enumerations of monomer-dimer configurations.

$M \times N$ lattice	$G_{MD}(1,1)$ given by (2)	$G_{MD}^{\text{free}}(1,1)$ given by (3)
5×5	3,190	1,536
5×7	53,010	24,150
7×5	56,434	24,150
7×7	3,118,178	1,204,224
7×9	171,527,426	57,961,134
9×7	165,771,810	57,961,134
9×9	29,845,632,402	8,921,088,000

To derive (2) we consider first the *close-packed* dimer problem on a related lattice $\mathcal{L}'$ constructed from $\mathcal{L}$ by connecting all M sites on one boundary to a single new site S as shown in Fig. 1(b). Dimers connecting boundary sites to S all carry weight 1. It is of interest to note that the lattice $\mathcal{L}'$ is self-dual and that the lattice has been considered previously by Lu and Wu [9] in the context of Ising partition function zeroes.

Denote the generating function of close-packed dimers on $\mathcal{L}'$ by $G_D(\mathcal{L}'; x, y)$. Since in a close-packed configuration S must be covered by a dimer (of weight 1), and the dimer must end at one of the M equivalent boundary sites which can be regarded as being occupied by a monomer on $\mathcal{L}$, there exists a correspondence between dimer configurations on $\mathcal{L}'$ and monomer-dimer configurations on $\mathcal{L}$. We are led to the identity

$$G_{MD}(x, y) = 2\, G_D(\mathcal{L}'; x, y), \tag{4}$$

where the extra factor 2 comes from the fact that there are 2 boundaries on a cylinder.

To evaluate $G_D(\mathcal{L}'; x, y)$ we introduce the lattice $\mathcal{L}''$ shown in Fig. 1(c) where S is connected to only one boundary site. Denote the generating function of close-packed dimers on $\mathcal{L}''$ by $G_D(\mathcal{L}''; x, y)$. It is clear that we have the further identity

$$G_D(\mathcal{L}'; x, y) = M\, G_D(\mathcal{L}''; x, y). \tag{5}$$

It remains to evaluate $G_D(\mathcal{L}''; x, y)$. But this is the problem solved in [7, 8].

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In the analysis given in [7], close-packed dimers on a lattice similar to $\mathcal{L}''$ are enumerated using the Kasteleyn approach [1]. Since our procedure follows closely that discussed in [7], we give an outline and highlight the difference.

Orient edges of $\mathcal{L}''$ and associate a phase factor i to all x edges as shown in Fig. 1(c). The only thing new from [7] is that we need to ascertain signs of all terms in the Pfaffian are the same. However, it can be shown [10, 11] that this always is the case for $M = $ odd. Then the desired generating function $G_D(\mathcal{L}''; x, y)$ is given in terms of the Pfaffian of a matrix A' [8],

$$i^{(M-1)/2} G_D(\mathcal{L}''; x, y) = \mathrm{Pf}(A') = \sqrt{\det A'}. \tag{6}$$

Here A' is the antisymmetric Kasteleyn matrix of dimension $(MN+1) \times (MN+1)$ for the lattice $\mathcal{L}''$ explicitly given by

$$A' = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & & & & & & & \\ \vdots & & & & & & & \\ 0 & & & & & & & \\ -1 & & & & A & & & \\ 0 & & & & & & & \\ \vdots & & & & & & & \\ 0 & & & & & & & \end{pmatrix}, \tag{7}$$

where A is the Kasteleyn matrix of dimension $MN \times MN$ for $\mathcal{L}$. The position of the elements ± 1 in the first row and column is that of the site $\{m, 1\}$ connected to S (see below). Explicitly, A is given by

$$A = ix S_M \otimes I_N + y I_M \otimes T_N, \tag{8}$$

with I_M is the $M \times M$ identity matrix, S_M is the *periodic* $M \times M$ matrix

$$S_M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}, \tag{9}$$

and T_N is the $N \times N$ matrix

$$
T_N = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
-1 & 0 & 1 & \cdots & 0 & 0 \\
0 & -1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & \cdots & -1 & 0
\end{pmatrix}.
\tag{10}
$$

Note that we have T_M instead of S_M in the corresponding expression in [7].

Label elements of A by $\{m, n; m', n'\}$, where (m, n) specifies the column and row of the position a site. The determinant of the Kasteleyn matrix A' can be computed by Laplace expanding along the first row and first column leading to

$$
\det A' = C(A; \{m, 1; m, 1\}),
\tag{11}
$$

where $C(A; \{m, 1; m, 1\})$ is the cofactor of the $\{m, 1; m, 1\}$ element of A, and we have specified the site connecting to S in Fig. 1(c) as $\{m, 1\}$.

Since the cofactor $C(A; \{m, 1; m, 1\})$ is proportional to the product of the nonzero eigenvalues of the matrix A, we need to determine the eigenvalues of A. This is done in the next section.

III. EIGENVALUES OF THE KASTELEYN MATRIX A

The matrix S_M can be diagonalized by the similarity transformation

$$
V_M^{-1} S_M V_M = \Omega_M,
$$

where V_M and its inverse V_M^{-1} are $M \times M$ unitary matrices with elements

$$
\begin{aligned}
V_M(m_1, m_2) &= \frac{1}{\sqrt{M}} e^{i2m_1 m_2 \pi/M}, \\
V_M^{-1}(m_1, m_2) &= \frac{1}{\sqrt{M}} e^{-i2m_1 m_2 \pi/M}, \quad 1 \le \{m_1, m_2\} \le M,
\end{aligned}
\tag{12}
$$

and Ω_M is an $M \times M$ diagonal matrix with the eigenvalues ω_m of S_M as entries,

$$
\omega_m = 2i \sin \frac{2m\pi}{M}, \qquad 1 \le m \le M.
$$

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Similarly, as in [7], the matrix T_N is diagonalized by the similarity transformation

$$U_N^{-1} T_N U_N = \Gamma_N,$$

where U_N and its inverse U_N^{-1} are $N \times N$ unitary matrices with elements

$$U_N(n_1, n_2) = \sqrt{\frac{2}{N+1}} \, i^{n_1} \sin\left(\frac{n_1 n_2 \pi}{N+1}\right),$$
$$U_N^{-1}(n_1, n_2) = \sqrt{\frac{2}{N+1}} \, i^{-n_2} \sin\left(\frac{n_1 n_2 \pi}{N+1}\right) \tag{13}$$

for $1 \le \{n_1, n_2\} \le N$, and Γ_N is an $N \times N$ diagonal matrix having eigenvalues γ_n of T_N as entries,

$$\gamma_n = 2i \cos\frac{n\pi}{N+1}, \qquad 1 \le n \le N.$$

Thus, the $MN \times MN$ matrix A can be diagonalized by the similarity transformation generated by $U_{MN} = V_M \otimes U_N$, leading to

$$U_{MN}^{-1} A U_{MN} = \Lambda_{MN}, \tag{14}$$

where Λ_{MN} is an $MN \times MN$ diagonal matrix having eigenvalues $\lambda_{m,n}$ of A as entries,

$$\lambda_{m,n} = 2i \left(ix \sin\frac{2m\pi}{M} + y \cos\frac{n\pi}{N+1} \right), \quad 1 \le m \le M, \quad 1 \le n \le N.$$

Note that $\lambda_{m,n}$ vanishes at $m = M, n = (N+1)/2$. Elements of U_{MN} and its inverse U_{MN}^{-1} are

$$U_{MN}\{m_1, n_1; m_2, n_2\} = V_M(m_1, m_2) U_N(n_1, n_2)$$
$$U_{MN}^{-1}\{m_1, n_1; m_2, n_2\} = V_M^{-1}(m_1, m_2) U_N^{-1}(n_1, n_2). \tag{15}$$

Using the identities $\sin(2\pi - \theta) = -\sin\theta$ and $\cos(\pi - \theta) = -\cos\theta$, the product

$$P \equiv \prod_{m=1}^{M} \prod_{\substack{n=0 \\ (m,n)\neq(M,\frac{N+1}{2})}}^{N-1} \lambda_{m,n}, \tag{16}$$

where the product excludes the zero eigenvalue at $(m,n) = (M, \frac{N+1}{2})$, can be rearranged as

$$P = Q \prod_{m=1}^{\frac{M-1}{2}} \prod_{n=1}^{\frac{N-1}{2}} \left(4x^2 \sin^2\frac{2m\pi}{M} + 4y^2 \cos^2\frac{n\pi}{N+1} \right)^2, \tag{17}$$

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where the factor Q collects all factors with either $n = (N+1)/2$ or $m = M$, namely,

$$Q = \prod_{m=1}^{\frac{M-1}{2}} \left(-4x^2 \sin^2 \frac{2m\pi}{M} \right) \prod_{n=1}^{\frac{N-1}{2}} \left(4y^2 \cos^2 \frac{n\pi}{N+1} \right)$$

$$= (-1)^{(M-1)/2} \left(\frac{M(N+1)}{2} \right) x^{M-1} y^{N-1}, \tag{18}$$

after using the identities

$$\prod_{m=1}^{\frac{M-1}{2}} \left(4 \sin^2 \frac{2m\pi}{M} \right) = M, \quad \prod_{n=1}^{\frac{N-1}{2}} \left(4 \cos^2 \frac{n\pi}{N+1} \right) = \frac{N+1}{2}, \quad M, N \text{ odd}.$$

The expressions (17) and (18) apply to $M, N \geq 3$ and will be used in the next section.

IV.   EVALUATION OF THE GENERATING FUNCTION (1)

We now compute the generating function (1).

Combining (4)-(6) with (11), we obtain the following expression,

$$G_{MD}(x, y) = 2 M \, i^{(1-M)/2} \sqrt{C(A; \{m, 1; m, 1\})}. \tag{19}$$

where $C(A; \{m, 1; m, 1\})$ is the cofactor of the $(m, 1; m, 1)$ element of the matrix A.

The computation of cofactors of a singular matrix like A requires special attention since the matrix does not possess an inverse. The difficulty was resolved in [7] by perturbing the matrix A slightly rendering it non-singular to permit an inverse. By carrying out this analysis details of which we refer to [7], one finds the cofactor

$$C(A; \{m, n; m', n'\}) = \left[U_{MN} \left(m', n'; M, \frac{N+1}{2} \right) U_{MN}^{-1} \left(M, \frac{N+1}{2}; m, n \right) \right] P, \tag{20}$$

where U_{MN} is the matrix diagonalizing A. Note that the index $\{M, \frac{N+1}{2}\}$ is that of the zero eigenvalue.

Elements of U_{MN} and U_{MN}^{-1} are given in (15). After combining with (12) and (13), we obtain from (20)

$$C(A; \{m, n; m', n'\}) = \left[\frac{2 \, i^{n'-n}}{M(N+1)} \sin \frac{n\pi}{2} \sin \frac{n'\pi}{2} \right] P \tag{21}$$

valid for general m, n, m', n'.

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Finally, we combine (4)-(6) with (11) and (21) at $\{m' = m, n' = n = 1\}$, and arrive at the expression

$$G_{MD}(x, y) = 2\, M\, i^{(1-M)/2} \sqrt{\frac{2P}{M(N+1)}}. \tag{22}$$

This yields the generating function (2) given in Sec. II after substituting with P given by (17) and Q by (18). We note that the result is independent of m as it should.

Then, with the help of the relations

$$\prod_{n=1}^{\frac{N-1}{2}} F\left(\cos^2 \frac{n\pi}{N+1}\right) = \prod_{n=1}^{\frac{N-1}{2}} F\left(\sin^2 \frac{n\pi}{N+1}\right) \quad \text{and} \quad \prod_{m=1}^{\frac{M-1}{2}} F\left(\sin^2 \frac{2m\pi}{M}\right) = \prod_{m=1}^{\frac{M-1}{2}} F\left(\sin^2 \frac{m\pi}{M}\right),$$

valid for any function $F(\cdot)$, the generating function (2) can be written in the equivalent form,

$$G_{MD}(x, y) = 2Mx^{(M-1)/2}y^{(N-1)/2} \prod_{m=1}^{\frac{M-1}{2}} \prod_{n=1}^{\frac{N-1}{2}} \left(4x^2 \sin^2 \frac{m\pi}{M} + 4y^2 \sin^2 \frac{n\pi}{N+1}\right), \quad M, N = \text{odd}. \tag{23}$$

It is convenient at this point to introduce a function

$$\begin{aligned} H(z; M, N) &\equiv \left[\prod_{m=0}^{M-1} \prod_{\substack{n=0 \\ (m,n)\neq(0,0)}}^{N-1} \left(4z^2 \sin^2 \frac{m\pi}{M} + 4\sin^2 \frac{n\pi}{N}\right)\right]^{1/2} \\ &= z^{MN-1} H(1/z; N, M), \qquad \text{any } M, N > 1. \end{aligned} \tag{24}$$

It will be shown in Appendix A that we have

$$G_{MD}(x, y) = R_{M,N}(y, z)\, \sqrt{H(z; M, N+1)}, \qquad M, N = \text{odd}, \tag{25}$$

where $z = x/y$ and

$$\begin{aligned} [R_{M,N}(y, z)]^2 &= \frac{4My^{MN-1}}{(N+1)z^M S_M(z)}, \\ S_M(z) &= \sinh\left(M\sinh^{-1}(1/z)\right). \end{aligned}$$

The advantage of using (25) instead of (23) for the generating function is that the factor $R_{M,N}(y, z)$ sorts out major contributions in the asymptotic expansions of the free energy (30) and (31) discussed below.

Two equivalent expressions of $H(z; M, N + 1)$ can be obtained by taking one of the products in (24) in a closed form. Taking the product over n, we obtain

$$H(z; M, N+1) = (N+1) \prod_{m=1}^{M-1} 2\sinh\left[(N+1)\omega_z\left(\tfrac{m\pi}{M}\right)\right] \tag{26}$$

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where

$$\omega_z(k) = \sinh^{-1}(z \sin k) \tag{27}$$

is the lattice dispersion relation, and we have used the identities (A2) and (A4).

Similarly, taking the product over m and making use of (A4) and the equivalence (24), we obtain

$$H(z; M, N+1) = Mz^{M(N+1)-1} \prod_{n=1}^{N} 2\sinh\left[M\omega_{1/z}\left(\tfrac{n\pi}{N+1}\right)\right]. \tag{28}$$

V. FINITE-SIZE ANALYSIS AND ASYMPTOTIC EXPANSIONS

Define the "free energy" of the monomer-dimer system as

$$\begin{aligned}
F_{M,N}(x,y) &= -\ln G_{MD}(x,y) \\
&= -\ln R_{M,N}(y,z) - \frac{1}{2}\ln H(z; M, N+1),
\end{aligned} \tag{29}$$

where we have made use of (24). We note that other than an overall factor $\left[4\sin^2(\alpha\pi/M) + 4\sin^2(\beta\pi/N)\right]$, the function $H(z; M, N+1)$ is the special case of $\alpha = \beta = 0$ of a more generally defined function $Z_{\alpha,\beta}(z; M, N+1)$ introduced, and analyzed in details in [12, 13]. This permits us to use results of [12, 13] to write down a general expression for $F_{M,N}(x,y)$, which we shall not reproduce. Instead, we focus on the free energies

$$F_M = \lim_{N\to\infty} \frac{1}{N} F_{M,N}(x,y) \quad \text{and} \quad F_N = \lim_{M\to\infty} \frac{1}{M} F_{M,N}(x,y)$$

of infinite "strips" and their asymptotic expansions.

The asymptotic expansions can be deduced by applying the Euler-MacLaurin summation identity to $\ln H(z; M, N+1)$. Using $H(z; M.N+1)$ given by (26) and (28), respectively, we obtain from (29) using (26) and (28), respectively,

$$\begin{aligned}
F_M &= -\frac{M}{2}\ln y - \frac{1}{2}\sum_{m=1}^{M-1} \omega_z\left(\tfrac{m\pi}{M}\right) \\
&= Mf_{\text{bulk}} + \sum_{p=1}^{\infty} \left(\frac{\pi}{M}\right)^{2p-1} \frac{d_{2p-2}(z)}{(2p-2)!}\left(\frac{B_{2p}}{2p}\right) \\
&= Mf_{\text{bulk}} + \frac{\pi z}{12}\left(\frac{1}{M}\right) + \cdots, \quad \text{(infinite length)}, \tag{30}
\end{aligned}$$

$$
\begin{aligned}
F_N &= -\frac{N}{2}\ln(yz) + \frac{1}{2}\sinh^{-1}(1/z) - \frac{1}{2}\sum_{n=1}^{N}\omega_{1/z}\!\left(\tfrac{n\pi}{N+1}\right) \\
&= Nf_{\text{bulk}} + 2f_{\text{surface}} + \sum_{p=1}^{\infty}\left(\frac{\pi}{N+1}\right)^{2p-1}\frac{d_{2p-2}(1/z)}{(2p-2)!}\left(\frac{B_{2p}}{2p}\right) \\
&= Nf_{\text{bulk}} + 2f_{\text{surface}} + \frac{\pi}{12z}\left(\frac{1}{N+1}\right) - \cdots, \quad \text{(infinite perimenter)}. \qquad (31)
\end{aligned}
$$

where

$$
\begin{aligned}
f_{\text{bulk}} &= -\frac{1}{2}\ln y - \frac{1}{2\pi}\int_0^\pi \omega_z(k)dk \\
&= -\frac{1}{2}\ln(yz) - \frac{1}{2\pi}\int_0^\pi \omega_{1/z}(k)dk, \\
f_{\text{surface}} &= \frac{1}{4}\sinh^{-1}(1/z) - \frac{1}{4\pi}\int_0^\pi \omega_{1/z}(k)dk,
\end{aligned}
$$

$d_{2p}(z)$ are the coefficients in the Taylor expansion

$$
\omega_z(k) = \sum_{p=0}^{\infty}\frac{d_{2p}(z)}{(2p)!}k^{2p+1},
$$

with $d_0(z) = z$, $d_2(z) = -z(1+z^2)/3$, $d_4(z) = z(1+z^2)(1+9z^2)/5, \cdots$, and $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42, \cdots$, are the Bernoulli numbers. The two equivalent expressions of f_{bulk} are obtained from (30) and (31), respectively. We remark that the equivalence of the two expressions is verified by the intriguing integral identity

$$
\frac{1}{\pi}\int_0^\pi \left[\sinh^{-1}(z\sin\theta) - \sinh^{-1}\!\left(\tfrac{1}{z}\sin\theta\right)\right]d\theta = \ln z \qquad (32)
$$

obtained by noting that the derivative of the left-hand side of (32) with respect to z reduces to $1/z$ after carrying out the integration.

The general theory of finite-size analysis [14–16] dictates that the free energy per unit length of a lattice model at criticality on an infinitely long strip of width $\mathcal{N}$ assumes the form [16]

$$
F_{\mathcal{N}} = \mathcal{N}f_{\text{bulk}} + f_{\text{surface}} + \frac{\Delta}{\mathcal{N}} + \cdots, \qquad (33)
$$

in an asymptotic expansion where f_{bulk} and f_{surface} are free energy densities of the order of O(1) and Δ is a constant. Unlike the free energy densities, the constant Δ is universal and its value is related to the central charge c in the logarithmic conformal field theory in a relation which depends on the boundary conditions in the transversal direction. Explicitly,

Δ is proportional to an effective central charge $c_{\text{eff}} = c - 24\,h_{\text{min}}$, where c is the central charge characterizing the universality class of the lattice model, as [14, 17]

$$\Delta = -\frac{\pi\zeta}{6}\,c_{\text{eff}} = -\frac{\pi\zeta}{6}\left(c - 24\,h_{\text{min}}\right) \quad \text{on a cylinder of infinite length,} \tag{34}$$

$$\Delta = -\frac{\pi\zeta}{24}\,c_{\text{eff}} = -\frac{\pi\zeta}{24}\left(c - 24\,h_{\text{min}}\right) \quad \text{on a cylinder of infinite perimeter,} \tag{35}$$

where the number h_{min} is the smallest conformal weight in the spectrum of the Hamiltonian with the given boundary conditions and ζ is an anisotropy factor. In our case we find from (30) and (31) that $\zeta = z$ and $1/z$, and $\Delta = \pi z/12$ and $\pi/12z$, respectively, in (34) and (35).

To retain the characteristics of a monomer on the surface, we consider a cylinder of infinite perimeter in a geometry which retains two surfaces. Therefore we use (35) and (31), or F_N, for which the boundary condition in the transverse direction is free (open) boundaries. It is known [18] that for free (open) boundaries $h_{\text{min}} = 0$. Hence we deduce the central charges

$$c = c_{\text{eff}} = -2. \tag{36}$$

On the other hand, if one uses (30), or F_M, the system is an infinitely long cylinder with a perimeter M. The two physical boundaries of the lattice are located at infinity so the existence of a monomer on the boundary is immaterial. The situation reduces to that of a pure dimer problem studied in [17]. For $M = $ odd we are considering, the analysis of [17] also gives $\Delta = \pi\zeta/12$ as in (31). However, for M odd, the boundary in the transverse direction is "frustrated" requiring special attention. It is argued in [17] that in this case one should use (35) with $h_{\text{min}} = 0$. This again leads to the same central charges (36).

We remark that the $c = -2$ central charge has been reported previously [6] in the solution (3) of a single monomer on the surface of a rectangular net with free (open) boundaries.

VI.   SUMMARY

We have derived the closed-form expression of the monomer-dimer generating function for a non-bipartite rectangular lattice under cylindrical boundary conditions with a single monomer confined to reside on the boundary. We have also carried out a finite-size analysis of the free energy. Asymptotic expansions of the free energy of strips of infinite lengths in the periodic and free (open) directions are obtained using the Euler-MacLaurin summation formula. We find the central charge in the framework of the logarithmic conformal field theory to be $c = -2$.

VII.   ACKNOWLEDGMENTS

Appendix A

In this appendix we establish the expression (25) for the generating function.

First, we rewrite the generating function (23) as

$$G_{MD}(x,y) = 2M z^{\frac{M-1}{2}} y^{\frac{MN-1}{2}} \prod_{m=1}^{\frac{M-1}{2}} \prod_{n=1}^{\frac{N-1}{2}} g(m,n), \tag{A1}$$

where

$$g(m,n) \equiv 4\left(z^2 \sin^2 \frac{m\pi}{M} + \sin^2 \frac{n\pi}{N+1} \right).$$

To extend the limits of the products in (A1) to $M-1$ and N as in (24), we note

$$\prod_{\substack{m=0 \\ (m,n)\neq(0,0)}}^{M-1} \prod_{n=0}^{N} g(m,n) = C_1\, C_2, C_3 \left[\prod_{m=1}^{\frac{M-1}{2}} \prod_{n=1}^{\frac{N-1}{2}} g(m,n) \right]^4 \qquad M, N = \text{odd},$$

where C_1, C_2, C_3 collect respective products for $m = 0$, $n = 0$, and $\{m \neq 0, n = (N+1)/2\}$. Namely, for M, N odd,

$$C_1 = \prod_{n=1}^{N} g(0,n) = \prod_{n=1}^{N} \left(4\sin^2 \frac{n\pi}{N+1} \right) = (N+1)^2, \quad N \geq 1, \tag{A2}$$

$$C_2 = \prod_{m=1}^{M-1} g(m,0) = \prod_{m=1}^{M-1} \left(4z^2 \sin^2 \frac{m\pi}{M} \right) = M^2 z^{2(M-1)}, \quad M > 1, \tag{A3}$$

$$C_3 = \prod_{m=1}^{M-1} g\left(m, \frac{N+1}{2}\right) = z^{2M} \sinh^2 \left[M \sinh^{-1} \left(\frac{1}{z}\right) \right], \quad M > 1, \tag{A4}$$

where the product (A4) is a special cases of the identity [19]

$$\prod_{m=0}^{M-1} \left(4\sinh^2\theta + 4\sin^2\frac{m\pi}{M} \right) = 4\sinh^2(M\theta), \quad M \geq 1. \tag{A5}$$

Combining these results, the generating function (A1) reduces to (25).

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[11] Technically, this is because no transposition polygon obtained by superimposing two dimer configurations, which is always of even length, can loop around the cylinder for $M = $ odd. It follows that we do not need to worry about the periodic boundary condition, and the signs of all terms in the Pfaffian are the same as determined in [7]. Furthermore, since the monomer resides on the boundary, we do not need to worry about transposition polygons looping around the monomer which, if existing, change the sign of some terms.

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[19] See, for example, I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, 1994, formula 1.394 with $x = e^{\theta}, y = e^{-\theta}$.